Some properties of electromagnetic waves near the interface of dielectric media

1991 J. Phys. A: Math. Gen. 24 L323

(http://iopscience.iop.org/0305-4470/24/7/002)

J. Phys. A: Math. Gen. 24 (1991) L323-L327. Printed in the UK

LETTER TO THE EDITOR

Some properties of electromagnetic waves near the interface of dielectric media

Liu Xin Ya

Department of Physics, Jiangxi University, Nanchang, Jiangxi 330047, People's Republic of China

Received 24 October 1990, in final form 17 January 1991

Abstract. The problem of reflection and refraction of electromagnetic waves on the interface of dielectric media is dealt with according to the generalized variational principle and some properties of the electromagnetic waves near the interface are deduced from the transformation properties of the constrained system under the transformation of coordinates. These lead to the equation of motion of the centre of energy which shows that the transverse shift is in existence.

Since the transverse shift (TS) of a totally reflected light beam was predicted in the 1950s a number of research papers on the TS phenomenon have been published [1-4], of which the theoretical explanations are based on Maxwell's equations or conservation laws of electromagnetic fields and there are different opinions on both the conditions of the existence of this effect and its value. We think it is necessary to carry out further research to look for a more reasonable approach to TS. In this letter we try to deal with the problem of reflection and refraction of electromagnetic waves on the interface of dielectric media according to the generalized variational principle and to deduce some properties of the electromagnetic waves from the transformation properties of the constrained system under the transformation of coordinates.

For simplicity, let us consider a spacetime restricted quasimonochromatic packet of electromagnetic waves which impinges upon a interface from medium 1 and is partly refracted into medium 2. The independent boundary conditions of the electromagnetic waves on the interface are the following equations:

$$\boldsymbol{h}x(\boldsymbol{E}_1-\boldsymbol{E}_2)=0 \qquad (1)$$

$$\boldsymbol{n}x(\boldsymbol{H}_1-\boldsymbol{H}_2)=0 \qquad (2)$$

where $\boldsymbol{n}$ stands fot the normal unit vector of the interface directing to medium 1. The right-hand Cartesian coordinates can be set up as the following: the x_3 axis is the direction of vector $\boldsymbol{n}$ and the x_1 axis and x_2 axis are on the interface. The x_1 axis is in the direction of the normal of the incident plane according to the relations between field and potential and by introducing the four-potential $A^{\alpha}(A, \mathrm{i}\phi)$, equations (1) and (2) can be rewritten as the following constraint conditions (take $c=1$):

$$G_1=A^4_{1,1}-A^1_{1,4}-A^4_{2,1}+A^1_{2,4}=0 \qquad (3)$$

$$G_2=A^4_{1,2}-A^2_{1,4}-A^4_{2,2}+A^2_{2,4}=0 \qquad (4)$$

$$G_3 = \frac{1}{\mu_1} A^3_{1,2} - \frac{1}{\mu_1} A^2_{1,3} - \frac{1}{\mu_2} A^3_{2,2} + \frac{1}{\mu_2} A^2_{2,3} = 0 \tag{5}$$

$$G_4 = \frac{1}{\mu_1} A^1_{1,3} - \frac{1}{\mu_1} A^3_{1,1} - \frac{1}{\mu_2} A^1_{2,3} + \frac{1}{\mu_2} A^3_{2,1} = 0 \tag{6}$$

where $A^\alpha_{j,\nu} = \partial A^\alpha_j / \partial x_\nu$ ($\alpha = 1, 2, 3, 4$; $j = 1, 2$; $\nu = 1, 2, 3, 4$; $x_\nu = (\boldsymbol{x}, \mathrm{i}t)$).

Now let us consider the generalized variation of the above electromagnetic system. It is known that the Lagrangian of a free electromagnetic field may be chosen as

$$\mathscr{L} = -\tfrac{1}{2} Z^\alpha_{,\nu} A^\alpha_{,\nu}. \tag{7}$$

According to the generalized variational principle, the generalized variation can be turned into the natural one in terms of Lagrange multipliers and the generalized action can be written as

$$I^* = \int (\mathscr{L} + \lambda_r G_r)\, \mathrm{d}^4 x \qquad (r = 1, 2, 3, 4) \tag{8}$$

where G_r are the constraint conditions, and $\lambda_r(x)$ are the Lagrangian multipliers which vanish at the places where no constraint exists. In the present case λ_r vanish except on the interface. The equations of motion of the electromagnetic system are given by $\delta I^* = 0$. In the process of operation, A^α and λ_r are regarded as independent variables and three-dimensional space is divided into the subspaces v_j ($j = 1, 2$) by the interface. After using the Gauss theorem it follows that

$$\delta I^* = \int \left(\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,3}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,3}} \right) \delta A^\alpha_j \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_4 - \int \left(\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,\nu}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,\nu}} \right)_{,\nu} \delta A^\alpha_j \, \mathrm{d}^4 x$$
$$+ \int \left(\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,4}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,4}} \right) \delta A^\alpha_j \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \Big|^{\mathrm{i}t_2}_{\mathrm{i}t_1} + \int G_r \delta \lambda_r \tag{9}$$

where the index j indicates the quantities belonging to v_j. From $\delta I^* = 0$ and the independence of A^α_j and λ_r we have

$$\left(\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,\nu}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,\nu}} \right)_{,\nu} = 0 \qquad x_3 > 0 \text{ or } x_3 < 0 \tag{10}$$

$$G_r = 0 \qquad x_3 = 0 \tag{11}$$

$$\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,3}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,3}} = 0 \qquad x_3 = 0 \tag{12}$$

$$\frac{\partial \mathscr{L}^{(j)}}{\partial A^\alpha_{j,4}} + \lambda_r \frac{\partial G_r}{\partial A^\alpha_{j,4}} = 0 \tag{13}$$

in which (10) is the equation of motion of the electromagnetic field, and (11) is the constraint condition and (12) and (13) are the equations identifying the Lagrange multipliers. In order to identify λ_r we put (3)–(6) into (12) and (13), and obtain

$$\lambda_1 = -A^1_{1,4} = A^2_{2,4} \qquad \lambda_2 = -A^2_{1,4} = A^2_{2,4}$$

$$\lambda_3 = -\mu_1 A^2_{1,3} = \mu_2 A^2_{2,3} \qquad \lambda_4 = \mu_1 A^1_{1,3} = -\mu_2 A^1_{2,3}$$

which are the values of λ_r on the interface and outside $\lambda_r = 0$.

Now let us discuss the transformation properties of the above constrained electromagnetic system under the transformation of coordinates. The changes of spacetime points and potential functions under infinitesimal transformation can be written as

$$\delta x_\mu = x'_\mu - x_\mu = \xi_{\mu(s)}\omega_s \tag{14}$$

$$\bar{\delta} A^\alpha = A^{\alpha'}(x') - A^\alpha(x) = \zeta^\alpha_{(s)}\omega_s \tag{15}$$

$$\delta A^\alpha = A^{\alpha'}(x) - A^\alpha(x) = \eta^\alpha_{(s)}\omega_s \tag{16}$$

in which ω_s $(s = 1, 2, \ldots, l)$ are the parameters of Lie group of the transformation. It is assumed in field theory that the functional formulation of Lagrangian is invariant and that the actions at the same physical point are equal under the transformation, i.e.

$$\mathscr{L}(A^{\alpha'}_{,\mu})\,\mathrm{d}^4x' = \mathscr{L}(A^\alpha_{,\mu})\,\mathrm{d}^4x \tag{17}$$

where d^4x' and d^4x stand for the same volume element in four-dimensional space under differential coordinates and satisfy $\mathrm{d}^4x' = (1 + \partial_\mu \delta x_\mu)\,\mathrm{d}^4x$. Then from (17) we can obtain

$$\frac{\partial \mathscr{L}}{\partial A^\alpha_{,\mu}}\,\delta A^\alpha_{,\mu} + \mathscr{L}\frac{\partial \delta x_\mu}{\partial x_\mu} = 0. \tag{18}$$

Because the Lagrangian does not explicitly depend on x_μ, equation (18) may be converted into

$$\int \left(-\partial_\mu \frac{\partial \mathscr{L}}{\partial A^\alpha_{,\mu}}\right)\delta A^\alpha\,\mathrm{d}^4x + \int \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial A^\alpha_{,\mu}}\,\delta A^\alpha + \mathscr{L}\delta x_\mu\right)\mathrm{d}^4x = 0. \tag{19}$$

The change of the constraint conditions under the infinitesimal transformation is $\delta G_r = (\partial G_r/\partial A^\alpha_{,\mu})\delta A^\alpha_{,\mu}$. Multiplying δG_r by λ_r and integrating the product $\lambda_r \delta G_r$, we have

$$\int\left[\left(-\partial_\mu \frac{\partial(\lambda_r G_r)}{\partial A^\alpha_{,\mu}}\right)\delta A^\alpha + \partial_\mu\left(\frac{\partial(\lambda_r G_r)}{\partial A^\alpha_{,\mu}}\,\delta A^\alpha\right)\right]\mathrm{d}^4x = \int \lambda_r \delta G_r\,\mathrm{d}^4x. \tag{20}$$

Adding (20) to (19) and assuming that the motion of the electromagnetic field obeys Euler–Lagrange equations and that the integrated region can be arbitrary, we can obtain

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial A^\alpha_{,\mu}}\,\delta A^\alpha + \mathscr{L}\delta x_\mu\right) = -\partial_\mu\left(\lambda_r \frac{\partial G_r}{\partial A^\alpha_{,\mu}}\right)\delta A^\alpha \tag{21}$$

which is called the equation for the transformation properties of the electromagnetic system under the transformation of coordinates [5].

Next let us consider two usual transformations.

(I) The transformation of parallel translation: $\delta x_\mu = \xi_\mu$, $\bar{\delta} A^\alpha = 0$, $\delta A^\alpha = -A^\alpha_{,\mu}\xi_\mu$. In this case (21) becomes

$$\partial_\nu T_{\mu\nu} = H_\mu \tag{22}$$

where

$$T_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial A^\alpha_{,\nu}}\,A^\alpha_{,\mu} - \mathscr{L}\delta_{\mu\nu} \tag{23}$$

$$H_\mu = -\partial_\nu\left(\lambda_r \frac{\partial G_r}{\partial A^\alpha_{,\nu}}\right)A^\alpha_{,\mu}. \tag{24}$$

The $T_{\mu\nu}$ is the tensor of the energy-momentum density of the electromagnetic waves. The integration of (22) over three-dimensional space becomes

$$\partial_4 \int T_{\mu 4}\, \mathrm{d}c = -\int_{x_3=0} \left(T^{(1)}_{\mu 3} + \lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{1,3}} A^{\alpha}_{1,3} \right) \mathrm{d}x_1\, \mathrm{d}x_2 - \int_{x_3=0} \left(T^{(2)}_{\mu 3} + \lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{2,3}} A^{\alpha}_{2,3} \right) \mathrm{d}x_1\, \mathrm{d}x_2 - \Delta^{(1)}_{\mu} - \Delta^{(2)}_{\mu} \tag{25}$$

where

$$\Delta^{(j)}_{\mu} = \int_{\nu_j} \left[\partial_4 \left(\lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{j,4}} A^{\alpha}_{j,4} \right) - \lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{j,\nu}} A^{\alpha}_{j,\mu\nu} \right] \mathrm{d}v. \tag{26}$$

Substituting (23) and (22) into (25), we have

$$\partial_4 \int T_{\mu 4}\, \mathrm{d}v = \delta_{\mu 3} \left(\int_{x_3=0} \mathscr{L}^{(1)}\, \mathrm{d}x_1\, \mathrm{d}x_2 + \int_{x_3=0} \mathscr{L}^{(2)}\, \mathrm{d}x_1\, \mathrm{d}x_2 \right) - \Delta^{(1)}_{\mu} - \Delta^{(2)}_{\mu}. \tag{27}$$

Because $\int T_{\mu 4}\, \mathrm{d}v = (\boldsymbol{P}, iH)$ is the four-momentum, equation (27) implies that the components P_1, P_2 and energy H are conserved respectively if the interface is infinitely thin (then $\Delta^{(j)}_{\mu} \to 0$).

(II) The Lorentz transformation: $\delta x_{\mu} = \varepsilon_{\mu\nu} x_{\nu}$, $\bar{\delta} A^{\alpha} = \frac{1}{2}\varepsilon_{\mu\nu} D^{\alpha\beta}_{\mu\nu} A^{\beta}$,

$$\alpha A^{\alpha} = \tfrac{1}{2}\varepsilon_{\mu\nu}(D^{\alpha\beta}_{\mu\nu} A^{\beta} + x_{\mu} A^{\alpha}_{,\nu} - x_{\nu} A^{\alpha}_{,\mu})$$

where $D^{\alpha\beta}_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}$ are the elements of a tensor representation of a Lorentz group. In this case (21) becomes

$$\partial_{\nu} J_{\rho\mu\nu} = H_{\rho\mu} \tag{28}$$

where

$$J_{\rho\mu\nu} = \frac{\partial \mathscr{L}}{\partial A^{\alpha}_{,\nu}} D^{\alpha\beta}_{\rho\mu} A^{\beta} + x_{\rho} T_{\mu\nu} - x_{\mu} T_{\rho\nu} \tag{29}$$

$$H_{\rho\mu} = -\partial_{\nu} \left(\lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{,\nu}} \right) (D^{\alpha\beta}_{\rho\mu} A^{\beta} + x_{\rho} A^{\alpha}_{,\mu} - x_{\mu} A^{\alpha}_{,\rho}). \tag{30}$$

$J_{\rho\mu\nu}$ is the tensor of the density of the angular momentum of the electromagnetic waves. The integration of (28) over three-dimensional space becomes

$$\partial_4 \int J_{\rho\mu 4}\, \mathrm{d}v = \int_{x_3=0} (x_{\rho}\delta_{\mu 3} - x_{\mu}\delta_{\rho 3}) \mathscr{L}^{(1)}\, \mathrm{d}x_1\, \mathrm{d}x_2 + \int_{x_3=0} (x_{\rho}\delta_{\mu 3} - x_{\mu}\delta_{\rho 3}) \mathscr{L}^{(2)}\, \mathrm{d}x_1\, \mathrm{d}x_2 - \Delta^{(1)}_{\rho\mu} - \Delta^{(2)}_{\rho\mu} \tag{31}$$

where

$$\Delta^{(j)}_{\rho\mu} = \int_{\nu_j} \left[\partial_4 \left(\lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{j,4}} (D^{\alpha\beta}_{\rho\mu} A^{\beta}_{j} + x_{\rho} A^{\alpha}_{j,\mu} - x_{\mu} A^{\alpha}_{j,\rho}) \right) - \lambda_r \frac{\partial G_r}{\partial A^{\alpha}_{j,\nu}} \partial_{\nu} (D^{\alpha\beta}_{\rho\mu} A^{\beta}_{j} + x_{\rho} A^{\alpha}_{j,\mu} - x_{\mu} A^{\alpha}_{j,\rho}) \right] \mathrm{d}v. \tag{32}$$

Because $M_1 = \int J_{234}\, \mathrm{d}v$, $M_2 = \int J_{314}\, \mathrm{d}v$, and $M_3 = \int J_{124}\, \mathrm{d}v$ are the components of the total angular momentum, M_3 is conserved if the interface is infinitely thin.

Let us further consider equation (31) to find the centre of energy of the reflected and refracted waves. We take $\rho = i$ $(i = 1, 2, 3)$, $\mu = 4$ and put the expressions of J_{i44} and $D_{i4}^{\alpha\beta}$ into (31), and then we can obtain

$$\partial_4 \int x_i T_{44}\, \mathrm{d}v = \partial_4 \int (A^i_{,4}A^4 - A^4_{,4}A^i + x_4 T_{i4})\, \mathrm{d}v$$
$$- \delta_{i3}x_4\left(\int_{x_3=0} \mathscr{L}^{(1)}\, \mathrm{d}x_1\, \mathrm{d}x_2 + \int_{x_3=0} \mathscr{L}^{(2)}\, \mathrm{d}x_1\, \mathrm{d}x_2\right) - \Delta^{(1)}_{i4} - \Delta^{(2)}_{i4}. \tag{33}$$

Let $X_i = (1/iH) \int x_i T_{44}\, \mathrm{d}v$ be the coordinates of the centre of energy; from (33) and (27) the equations of motion of the centre of energy are found as follows

$$H\frac{\mathrm{d}X_i}{\mathrm{d}t} - (\Delta^{(1)}_4 + \Delta^{(2)}_4)X_i$$
$$= P_i - (\Delta^{(1)}_i + \Delta^{(2)}_i)x_4 + \int (A^i_{,44}A^4 - A^4_{,44}A^i)\, \mathrm{d}v - \Delta^{(1)}_{i4} - \Delta^{(2)}_{i4} \tag{34}$$

which show that the centre of energy will shift along the x_1 axis provided that $\Delta^{(j)}_4$ or $\Delta^{(j)}_{i4}$ is in existence.

References

[1] Picht J 1955 *Abhandlungen der Deitschen Akademii der Wissenschaften zur Berlin, Klasse für Mathematik, Physik, und Technik Physik, und Technik 2*
[2] Fedorov F I 1955 *Dokl. Akad. Nauk, SSSR* **105** 465
[3] Schilling H 1965 *Ann. Phys., Lpz* **16** 122–34
[4] Fedoseyev V G 1988 *J. Phys. A: Math. Gen.* **21** 2045
[5] Li Zi Ping 1981 *Acta Phys. Sinica* **30** 1659 (in Chinese)